WAVY SPIRALS AND THEIR FRACTAL CONNECTION WITH CHIRPS

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Abstract. We study the fractal oscillatority of a class of real C^1 functions $x = x(t)$ near $t = \infty$. It is measured by *oscillatory* and *phase dimensions*, defined as box dimensions of the graph of $X(\tau) = x(\frac{1}{\tau})$ near $\tau = 0$ and the trajectory $(x, \dot{x})$ in $\mathbb{R}^2$, respectively, assuming that $(x, \dot{x})$ is a spiral converging to the origin. The relationship between these two dimensions has been established for a class of oscillatory functions using formulas for box dimensions of graphs of chirps and nonrectifiable wavy spirals, introduced in this paper. Also, the rectifiable chirps and spirals have been studied.

Keywords: Wavy spiral, chirp, box dimension, Minkowski content, rectifiability, oscillatory dimension, phase dimension
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1. Introduction

The chirp-like behavior of solutions of different types of second-order linear differential equations has been studied by Kwong, Pašić, Tanaka and Wong. Euler type equation is considered in [10], [11] and [18], Hartman-Wintner type equations in [7], half linear equations in [13], and for the Bessel equation see [14]. The fractal properties of spiral trajectories of dynamical systems in the phase plane have been studied by Žubrinić and Županović, see e.g. [19] and [21]. An interesting behavior of the box dimension of spiral trajectories has been found and related to the bifurcation of the system, in particular to the Hopf bifurcation. These results motivate us to study and analyze the connection between chirp-like functions and planar spiral trajectories in the phase plane. Our main results are obtained in Theorems 4 and 7. A specific type of spirals converging to the origin, but with nondecreasing radius function emerged in our study. We call them wavy spirals, see Definition 10.

We can summarize our results in the following way. An $(\alpha, 1)$-chirp-like function near the origin, described in Theorem 4, is connected with a spiral "similar" to $r = \varphi^{-\alpha}$, defined in polar coordinates. If $\alpha \in (0, 1)$ then the box dimension of this spiral is equal to $\frac{2}{\alpha+1}$, see Theorems 2 and 4(i), and for $\alpha > 1$ this spiral is rectifiable, see Theorem 4(ii).

Furthermore, we consider the opposite direction and generate a chirp from a given planar spiral. The obtained chirp is $(\alpha, 1)$-chirp-like function, $\alpha \in (0, 1)$, see Definition 8, and the box dimension of its graph is equal to $\frac{3-\alpha}{2}$, see Theorems 5 and 7. If the planar spiral is rectifiable and $\alpha > 1$, then the corresponding chirp-like function has rectifiable graph as well, see Theorem 8. For some applications of obtained results to differential equations and dynamical systems see [5] and [6]. This article has been motivated by Pašić, Žubrinić and Županović [15], showing the first results about chirps and spirals. Applications of our study include Liénard and Bessel equations.

Fractal analysis of bifurcations has been considered by Horvat Dmitrović [3], Li and Wu [8], Mardešić, Resman and Županović [9] and Resman [16].

2. Definitions

Let us introduce some definitions and notation. Given a bounded subset A of $\mathbb{R}^N$, we define the ε-neighborhood of A by $A_\varepsilon := \{y \in \mathbb{R}^N : d(y, A) < \varepsilon\}$, where $d(y, A)$ denotes the Euclidean distance from y to A. By *lower s-dimensional Minkowski content* of A, $s \geq 0$ we mean

$$\mathcal{M}_*^s(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}}$$

and analogously for the *upper s-dimensional Minkowski content* $\mathcal{M}^{*s}(A)$. If both these quantities coincide, the common value is called the *s-dimensional Minkowski content of A*, and is denoted by $\mathcal{M}^s(A)$. Now we can introduce the *lower and upper box dimensions* of A by

$$\underline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}_*^s(A) = 0\}$$

and analogously $\overline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}^{*s}(A) = 0\}$. If these two values coincide, we call it simply the box dimension of A, and denote it by $\dim_B A$.
If $0 < \mathcal{M}_*^d(A) \leq \mathcal{M}^{*d}(A) < \infty$ for some d, then we say that A is *Minkowski nondegenerate*. In this case obviously $d = \dim_B A$. In the case when lower or upper d-dimensional Minkowski contents of A are 0 or ∞, where $d = \dim_B A$, or $\underline{\dim}_B A < \overline{\dim}_B A$, we say that A is *degenerate*.
More details on these definitions can be seen in Falconer [2] and Tricot [17]. Some generalizations can be seen in [9].

Definition 1. Let $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, be a continuous function. We say that x is *oscillatory function* near $t = \infty$ if there exists a

sequence $t_k \to \infty$ such that $x(t_k) = 0$, and the functions $x|_{(t_k,t_{k+1})}$ alternately change sign for $k \in \mathbb{N}$.

Analogously, let $u : (0, t_0] \to \mathbb{R}$, $t_0 > 0$, be a continuous function. We say that u is *oscillatory function* near the origin if there exists a sequence s_k such that $s_k \searrow 0$ as $k \to \infty$, $u(s_k) = 0$ and restrictions $u|_{(s_{k+1},s_k)}$ alternately change sign for $k \in \mathbb{N}$.

Definition 2. (see Pašić [11]) Suppose that $v : I \to \mathbb{R}$, $I = (0, 1]$, is an oscillatory function near the origin, $d \in [1, 2)$. We say that v is *d-dimensional fractal oscillatory* near the origin if $\dim_B G(v) = d$ and $0 < \mathcal{M}^d_*(G(v)) \leq \mathcal{M}^{*d}(G(v)) < \infty$, where $G(v)$ denotes the graph of v.

Definition 3. Assume that $x : [t_0, \infty) \to \mathbb{R}$ is oscillatory near $t = \infty$. Let us define $X : (0, 1/t_0] \to \mathbb{R}$ by $X(\tau) = x(1/\tau)$. It is clear that $X(\tau)$ is oscillatory near the origin. We measure the rate of oscillatority of $x(t)$ near $t = \infty$ by the rate of oscillatority of $X(\tau)$ near $\tau = 0$. More precisely, the *oscillatory dimension* $\dim_{osc}(x)$ (near $t = \infty$) is defined as box dimension of the graph of $X(\tau)$ near $\tau = 0$:

$$\dim_{osc}(x) = \dim_B G(X),$$

provided the box dimension exists.

Definition 4. Assume now that x is of class C^1. We say that x is a *phase oscillatory* function if the following condition holds: the set $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ in the plane is a spiral converging to the origin.

Definition 5. By a *spiral* here we mean the graph of a function $r = f(\varphi)$, $\varphi \geq \varphi_1 > 0$, in polar coordinates, where

$$(1) \qquad \begin{cases} f : [\varphi_1, \infty) \to (0, \infty) \text{ is such that } f(\varphi) \to 0 \text{ as } \varphi \to \infty, \\ f \text{ is \textit{radially decreasing} (i.e., for any fixed } \varphi \geq \varphi_1 \\ \text{the function } \mathbb{N} \ni k \mapsto f(\varphi + 2k\pi) \text{ is decreasing).} \end{cases}$$

This definition appears in [19]. Depending on the context, by a spiral here we also mean the graph of a function $r = g(\varphi)$, $\varphi \leq \varphi_1' < 0$, in polar coordinates, such that the curve defined as the graph of $r = g(-\varphi)$, $\varphi \geq |\varphi_1'| > 0$, given in polar coordinates, satisfies (1). It is easy to see that a spiral defined by a function $g(\varphi)$ is a mirror image of the spiral defined by $g(-\varphi)$, with respect to the x-axis. We also say that a graph of a function $r = f(\varphi)$, $\varphi \geq \varphi_1 > 0$, defined in polar coordinates, is a *spiral near the origin* if there exists $\varphi_2 \geq \varphi_1$ such that the graph of the function $r = f(\varphi)$, $\varphi \geq \varphi_2$, viewed in polar coordinates, is a spiral.

Definition 6. The *phase dimension* $\dim_{ph}(x)$ of a function $x : [t_0, \infty) \to \mathbb{R}$ of class C^1 is defined as the box dimension of the corresponding planar curve $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$.

Oscillatory and phase dimensions are fractal dimensions, introduced in the study of chirp-like solutions of second order ODEs, see [15]. More about fractal dimensions in dynamics can be found in [22].

For two real functions $f(t)$ and $g(t)$ of a real variable we write $f(t) \simeq g(t)$ as $t \to 0$ (as $t \to \infty$) if there exist two positive constants C and D such that $C\, f(t) \leq g(t) \leq D\, f(t)$ for all t sufficiently close to $t = 0$ (for all t large enough). For a function $F : U \to V$, with $U, V \subset \mathbb{R}^2$, $V = F(U)$, we write $|F(x_1) - F(x_2)| \simeq |x_1 - x_2|$ if F is a bi-Lipschitz mapping, i.e., both F and F^{-1} are Lipschitzian.

Definition 7. We write $f(t) \sim g(t)$ if $f(t)/g(t) \to 1$ as $t \to 0$ (as $t \to \infty$). Also, if k is fixed positive integer, for two functions f and g of class C^k we write,

$$f(t) \sim_k g(t) \text{ as } t \to 0 \text{ (as } t \to \infty),$$

if $f^{(j)}(t) \sim g^{(j)}(t)$ as $t \to 0$ (as $t \to \infty$) for all $j = 0, 1, ..., k$.

For example, $\frac{(t-1)^{4-\alpha}}{t^4} \sim_3 t^{-\alpha}$ as $t \to \infty$, for $\alpha \in (0, 1)$.

Analogously, if k is fixed positive integer, for two functions f and g of class C^k we write

$$f(t) \simeq_k g(t) \text{ as } t \to 0 \text{ (as } t \to \infty),$$

if $f^{(j)}(t) \simeq g^{(j)}(t)$ as $t \to 0$ (as $t \to \infty$) for all $j = 0, 1, ..., k$.

We write $f(t) = O(g(t))$ as $t \to 0$ (as $t \to \infty$) if there exists a positive constant C such that $|f(t)| \leq C|g(t)|$ for all t sufficiently close to $t = 0$. (for all t large enough).

Definition 8. Functions of the form

$$y = P(x)\sin(Q(x)) \text{ or } y = P(x)\cos(Q(x)),$$

where $P(x) \simeq x^{\alpha}$, $Q(x) \simeq_1 x^{-\beta}$ as $x \to 0$, are called (α, β)-*chirp-like function* near $x = 0$.

3. Spirals generated by chirps

We study spirals generated by chirps in the sense of Theorem 4. To prove Theorem 4 about box dimension of a spiral generated by a chirp we need a new version of [19, Theorem 5]. Let us first recall [19, Theorem 5], cited here in a more condensed form, suitable for our purposes. The following theorem extends a result about box dimension of spiral from Dupain, Mendès France and Tricot, see [1] and [17].

Theorem 1. (see [19, Theorem 5]) *Let $f:[\varphi_1,\infty)\to(0,\infty)$ be a decreasing function of class C^2, such that $f(\varphi)\to 0$ as $\varphi\to\infty$. Let $\alpha\in(0,1)$. Assume that there exist positive constants $\underline{m}$, $\overline{m}$, M_1, M_2 and M_3 such that for all $\varphi\ge\varphi_1>0$,*

$$\underline{m}\,\varphi^{-\alpha}\le f(\varphi)\le\overline{m}\,\varphi^{-\alpha},$$
$$M_1\varphi^{-\alpha-1}\le|f'(\varphi)|\le M_2\varphi^{-\alpha-1},\quad |f''(\varphi)|\le M_3\varphi^{-\alpha}.$$

Let Γ be the graph of $r=f(\varphi)$ in polar coordinates. Then

$$\dim_B\Gamma=\frac{2}{1+\alpha}.$$

Now we provide a new version of Theorem 1.

Theorem 2. (Dimension of a piecewise smooth nonincreasing spiral) *Let $f:[\varphi_1,\infty)\to(0,\infty)$ be a nonincreasing and radially decreasing function, see (1), that is, a continuous, piecewise continuously differentiable function. We assume that the number of smooth pieces of f in $[\varphi_1,\overline{\varphi}_1]$ is finite, for any $\overline{\varphi}_1>\varphi_1$. Assume that there exist positive constants $\underline{m}$, $\overline{m}$, $\underline{a}$ and M such that for all $\varphi\ge\varphi_1$,*

$$\underline{m}\varphi^{-\alpha}\le f(\varphi)\le\overline{m}\varphi^{-\alpha},$$
$$\underline{a}\varphi^{-\alpha-1}\le f(\varphi)-f(\varphi+2\pi),$$

and for all φ where $f(\varphi)$ is differentiable,

$$|f'(\varphi)|\le M\varphi^{-\alpha-1}.$$

Let Γ be the graph of $r=f(\varphi)$ in polar coordinates. If $\alpha\in(0,1)$ then

$$\dim_B\Gamma=\frac{2}{1+\alpha}.$$

For the proof of Theorems 2 and 4 below, we need the following Lemma 1 that is a generalization of [19, Lemma 1] dealing with smooth spirals.

Lemma 1. (Excision property for piecewise smooth curves) *Let Γ be a piecewise smooth curve in $\mathbb{R}^2$, that is, Γ is the graph of a continuous and piecewise continuously differentiable function $h:[\varphi_1,\infty)\to\mathbb{R}^2$ (piecewise in the sense of the Theorem 2). Assume that $\underline{\dim}_B\Gamma>1$, $\Gamma_1:=h((\overline{\varphi}_1,\infty))$, for some fixed $\overline{\varphi}_1>\varphi_1$, and $h([\varphi_1,\overline{\varphi}_1])\bigcap\Gamma_1=\emptyset$. Then*

$$\underline{\dim}_B\Gamma_1=\underline{\dim}_B\Gamma,\quad\overline{\dim}_B\Gamma_1=\overline{\dim}_B\Gamma.$$

Proof. The proof is analogous to the proof of [19, Lemma 1], but with the following difference. Here, the curve $\Gamma_2:=\Gamma\backslash\Gamma_1=h([\varphi_1,\overline{\varphi}_1])$ is rectifiable due to piecewise rectifiability of h and due to the finite number of pieces in segment $(\varphi_1,\overline{\varphi}_1]$. The function h is piecewise rectifiable due to its piecewise smoothness and continuity. Also, by careful

examination of the proof of [19, Lemma 1], it follows that we can substitute the injectivity assumption on h with the weaker condition that $h([\varphi_1, \overline{\varphi}_1]) \bigcap \Gamma_1 = \emptyset$. (For more details see [19, Lemma 1].) □

Proof of Theorem 2. The proof is analogous to the proof of [19, Theorem 5], but using the new Lemma 1. □

Remark 1. Notice the difference between the assumptions of Theorem 1 and Theorem 2. In Theorem 1 the function f is decreasing and of class C^2. By careful examination of the proof of [19, Theorem 5], one can see that f being decreasing is used only in the sense of nonincreasing, that is, not strictly decreasing, hence in Theorem 1 we can assume that f is nonincreasing. Additional smoothness of f and additional conditions regarding constants M_1 and M_3 in Theorem 1 are used only in the calculations of Minkowski contents in [19, Theorem 5] which we exclude from our Theorem 2. Further reduction in smoothness of f from continuously differentiable to a piecewise continuously differentiable function can be found in Lemma 1.

Theorem 3 deals with a spiral Γ' described by $r = f(\varphi)$, where f is increasing on some parts, see Definitions 9 and 10. We call this new property of Γ' *spiral wavyness.*

Definition 9. Let $r : [t_0, \infty) \to (0, \infty)$ be a C^1 function. Assume that $r'(t_0) \le 0$. We say that $r = r(t)$ is a *wavy function* if the sequence (t_n) defined inductively by:

$$\begin{aligned} t_{2k+1} &:= \inf\{t : t > t_{2k}, r'(t) > 0\}, \quad k \in \mathbb{N}_0, \\ t_{2k+2} &:= \inf\{t : t > t_{2k+1}, r(t) = r(t_{2k+1})\}, \quad k \in \mathbb{N}_0, \end{aligned}$$

is well-defined, and satisfies the the *wavyness condition*:

$$(2) \qquad \left\{ \begin{array}{rl} \text{(i)} & \text{The sequence } (t_n) \text{ is increasing and } t_n \to \infty \text{ as } n \to \infty. \\ \text{(ii)} & \text{There exists } T > 0, \text{ such that for all } k \in \mathbb{N}_0 \\ & t_{2k+1} - t_{2k} < T \text{ and } t_{2k+2} - t_{2k+1} < \frac{5\pi}{12}. \\ \text{(iii)} & \text{There exists } C > 0, \text{ such that for all } k \in \mathbb{N}_0 \\ & \operatorname*{osc}_{t \in [t_{2k+1}, t_{2k+2}]} r(t) \le C t_{2k+1}^{-\alpha-2}, \end{array} \right.$$

where $\operatorname*{osc}_{t\in I} r(t) = \max_{t\in I} r(t) - \min_{t\in I} r(t)$.

Notice that $\min_{t\in[t_{2k+1},t_{2k+2}]} r(t) = r(t_{2k+1})$. Condition (i) means that the property of waveness of $r = r(\varphi)$ is global on the whole domain. Condition (ii) is a technical condition, see Remark 2. Condition (iii) is condition on decay rate on the sequence of oscillations of r on

$I_k = [t_{2k+1}, t_{2k+2}]$, $k \in \mathbb{N}_0$. Also, notice that condition $r'(t_0) \leq 0$ assures that t_1 is well-defined.

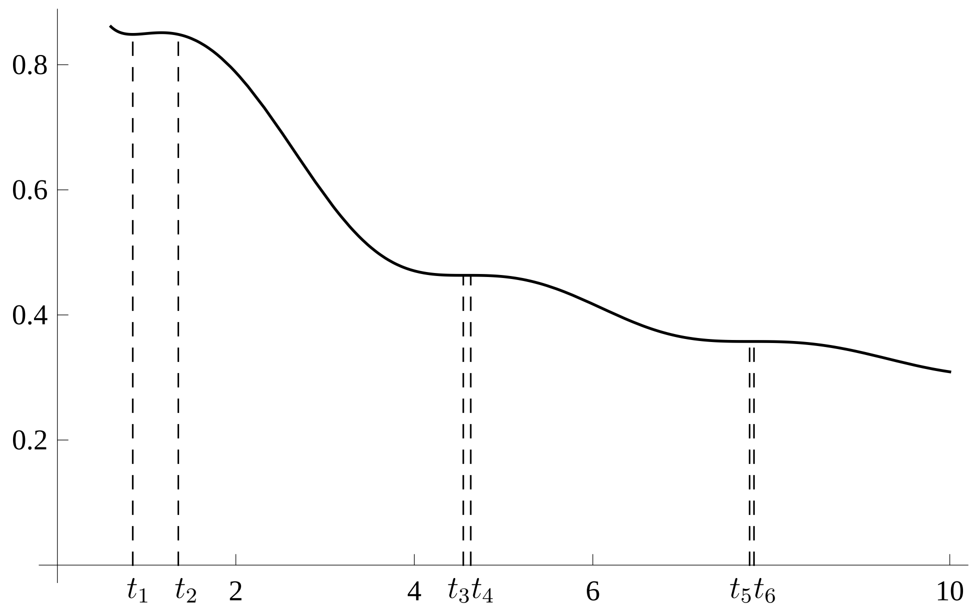

FIGURE 1. Function $r(t)$ for $p(t) = t^{-1/2}$, see Proposition 2, $t_0 = 0.6$. This is a wavy function, see Definition 9, with local minima at t_{2k+1}, $k = 0, 1, \ldots$

Definition 10. Let a spiral Γ', given in polar coordinates by $r = f(\varphi)$, where f is a given function. If there exists function $\varphi = \varphi(t)$ such that $r(t) = f(\varphi(t))$ is a wavy function, then we say Γ' is a *wavy spiral*.

Now, by using Theorem 2 and Lemma 1 we can prove Theorem 3.

Theorem 3. (Dimension of a wavy spiral) *Let $t_0 > 0$ and assume $r : [t_0, \infty) \to (0, \infty)$ is a function of class C^1 such that $r'(t_0) \leq 0$. Assume that $\varphi : [t_0, \infty) \to [\varphi_0, \infty)$ is an increasing function of class C^1 such that $\varphi(t_0) = \varphi_0 > 0$ and*

$$\varphi(t) - \varphi_0 = (t - t_0) + O(t^{-1}) \text{ as } t \to \infty. \tag{3}$$

Let $f : [\varphi_0, \infty) \to (0, \infty)$ be defined by $f(\varphi(t)) = r(t)$. Assume that Γ' is a spiral described in polar coordinates by $r = f(\varphi)$, satisfying (1). *Let $\alpha \in (0, 1)$ and assume that there exist positive constants $\underline{m}$, $\overline{m}$, $\underline{a}'$ and M such that for all $\varphi \geq \varphi_0$,*

$$\underline{m}\varphi^{-\alpha} \leq f(\varphi) \leq \overline{m}\varphi^{-\alpha}, \tag{4}$$

$$|f'(\varphi)| \leq M\varphi^{-\alpha-1}, \tag{5}$$

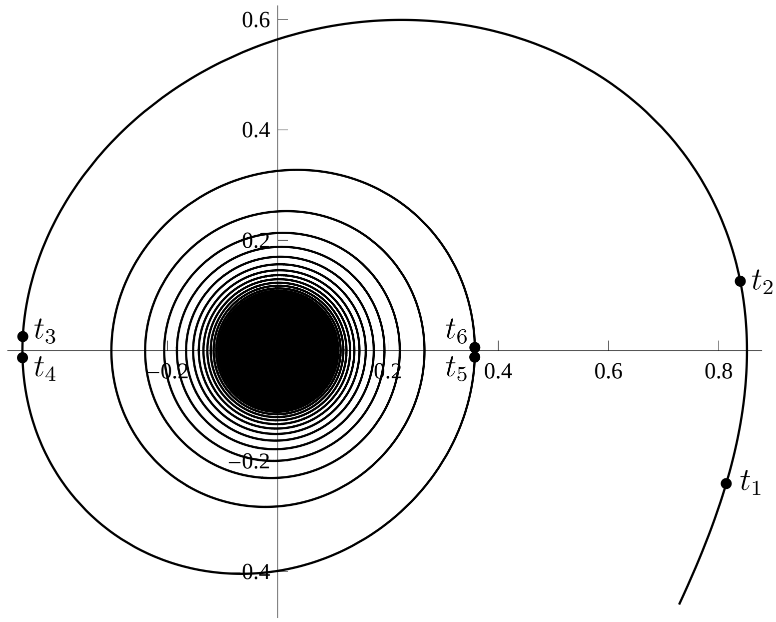

FIGURE 2. Spiral Γ' for $p(t) = t^{-1/2}$, see Proposition 2, $t_0 = 0.6$ is a wavy spiral, see Definition 10. Highlighted points correspond to parameters t_k, $k = 1, 2, \dots$

and for all $\triangle\varphi$, such that $\frac{\pi}{2} \leq \triangle\varphi \leq 2\pi + \frac{\pi}{2}$, there holds

$$\underline{a}'\varphi^{-\alpha-1} \leq f(\varphi) - f(\varphi + \triangle\varphi). \tag{6}$$

Assume that r is a wavy function. Then Γ' is a wavy spiral and

$$\dim_B \Gamma' = \frac{2}{1+\alpha}.$$

The proof of Theorem 3 is given in [6].

Remark 2. Notice that the upper bound of the difference $t_{2k+2} - t_{2k+1}$ could be increased to 2π to achieve greater generality of condition (2). The value $\frac{5\pi}{12}$ facilitates more elegant proof of Theorem 3. Lower value than $\frac{5\pi}{12}$ would be possible, although it would complicate the proof of Theorem 4 and not simplify the proof of Theorem 3. Notice that the bounds on $\triangle\varphi$ in condition (6) also indirectly depend on this value, see the proof of Theorem 3.

Now, Theorem 3 enables us to calculate the box dimension of a spiral generated by a chirp, which is one of the main results of this paper.

Theorem 4. (Chirp–spiral comparison) *Let $\alpha > 0$. Assume that X : $(0, 1/t_0] \to \mathbb{R}$, $t_0 > 0$, $X(\tau) = P(\tau)\sin 1/\tau$, where $P(\tau)$ is a positive*

function such that $P(\tau) \sim_3 \tau^\alpha$ *as* $\tau \to 0$. *Define* $x(t) = X(1/t)$ *and a continuous function* $\varphi(t)$ *by* $\tan\varphi(t) = \frac{\dot{x}(t)}{x(t)}$.

(i) *If* $\alpha \in (0,1)$ *then the planar curve* $\Gamma := \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ *generated by* X *is a wavy spiral* $r = f(\varphi)$, $\varphi \in (-\infty, -\varphi_0]$, $\varphi_0 > 0$, *near the origin. We have* $f(\varphi) \simeq |\varphi|^{-\alpha}$ *as* $\varphi \to -\infty$, *and*
$$\dim_{ph}(x) := \dim_B \Gamma = \frac{2}{1+\alpha}.$$
(ii) *If* $\alpha > 1$ *then the planar curve* $\Gamma := \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ *is a rectifiable wavy spiral near the origin.*

The proof of Theorem 4 consists of checking out the conditions of Theorem 3. The following two propositions make this checking easy.

Proposition 1. *Let* $\alpha > 0$ *and assume that* $P(\tau)$, $\tau \in (0, 1/t_0]$, $t_0 > 0$, *be such that* $P(\tau) \sim_3 \tau^\alpha$ *as* $\tau \to 0$. *Then* $p(t) := P(\frac{1}{t}) \sim_3 t^{-\alpha}$ *as* $t \to \infty$ *and vice versa. Furthermore, we have:*

$$\lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0, \ \lim_{t\to\infty} \frac{p''(t)}{p(t)} = 0, \tag{7}$$

$$-\frac{p(t)}{p'(t)} \sim \frac{t}{\alpha}, \quad -\frac{2p'(t)}{p''(t)} \sim \frac{2t}{\alpha+1} \text{ as } t \to \infty, \tag{8}$$

$$\sup_{t\in[t_0,\infty)} \left(-\frac{p(t)}{p'(t)}\right)' < \infty, \ \sup_{t\in[t_0,\infty)} \left(-\frac{2p'(t)}{p''(t)}\right)' < \infty. \tag{9}$$

The claims of Proposition 1 follow directly from the assumptions.

Proposition 2. *Let* $\alpha \in (0,1)$ *and*

$$r(t) = p(t)\sqrt{1 + \frac{p'^2(t)}{p^2(t)}\sin^2 t + \frac{p'(t)}{p(t)}\sin 2t}, \quad t \in [t_0, \infty), \ t_0 > 0,$$

where $p(t) \sim_1 t^{-\alpha}$ *as* $t \to \infty$.
Let $C \in \mathbb{R}$ *and assume that* $t(\varphi) = \varphi + C + O(\varphi^{-1})$ *as* $\varphi \to \infty$. *Let* $\triangle\varphi > 1$. *Then there exists constant* $k > 0$, *independent of* φ *and* $\triangle\varphi$, *such that for all* φ *sufficiently large it holds*
$$r(t(\varphi)) - r(t(\varphi + \triangle\varphi)) \geq k\varphi^{-\alpha-1}(1 + O(\varphi^{-1})).$$

The proof of Proposition 2 can be seen in the Appendix.

Proof of Theorem 4. (i)

Step 1. (Box dimension is invariant with respect to mirroring of a spiral.) We will prove an equivalent claim, that the planar curve $\Gamma' =$

$\{(x(t), -\dot{x}(t)) : t \in [t_0, \infty)\}$ is a wavy spiral defined by $r = f(\varphi)$, $\varphi \in [\varphi_0, \infty)$, near the origin, satisfying $f(\varphi) \simeq \varphi^{-\alpha}$ in polar coordinates, near the origin, and $\dim_B \Gamma' = \frac{2}{1+\alpha}$. It is easy to see that the curve Γ is a mirror image of the curve Γ', with respect to the x-axis, hence Γ is a wavy spiral. Reflecting with respect to the x-axis in the plane is an isometric map. As isometric map is bi-Lipschitzian and therefore it preserves box dimensions, see [2, p. 44], we see that $\dim_B \Gamma = \dim_B \Gamma' = \frac{2}{1+\alpha}$.

Step 2. (Checking condition (4).) From

$$\begin{aligned} x(t) &= p(t)\sin t, \\ \dot{x}(t) &= p'(t)\sin t + p(t)\cos t, \end{aligned}$$

where $p(t) := P(1/t)$, we compute

$$\tan\varphi(t) = -\frac{\dot{x}(t)}{x(t)} = -\frac{p'(t)}{p(t)} - \frac{1}{\tan t}. \tag{10}$$

By differentiating (10) we obtain

$$\frac{d\varphi}{dt}(t) = \cos^2\varphi(t)\left[\frac{p'^2(t) - p(t)p''(t)}{p^2(t)} + \frac{1}{\sin^2 t}\right]. \tag{11}$$

Using (10) again, we have

$$\cos^2\varphi(t) = \frac{1}{1+\tan^2\varphi(t)} = \frac{p^2(t)\sin^2 t}{p^2(t) + p'^2(t)\sin^2 t + 2p(t)p'(t)\sin t\cos t}.$$

Substituting into (11) and using (7) we get

$$\lim_{t\to\infty}\frac{d\varphi}{dt}(t) = 1. \tag{12}$$

From (12) it follows that $\varphi \simeq t$ as $t \to \infty$ and

$$r^2(t) = (x(t))^2 + (-\dot{x}(t))^2 = p^2(t) + p'^2(t)\sin^2 t + p(t)p'(t)\sin 2t \tag{13}$$

implies

$$f(\varphi(t)) = r(t) \simeq t^{-\alpha} \simeq \varphi^{-\alpha} \text{ as } t \to \infty. \tag{14}$$

Notice that from (13) it follows that the function r is of class C^2.

Step 3. (Checking condition (5).) On the other hand, differentiating (13) we obtain that

$$\begin{aligned} \frac{dr}{dt}(t) = \big[2p(t)p'(t)\cos^2 t &+ \frac{2p'^2(t) + p(t)p''(t)}{2}\sin 2t \\ &+ p'(t)p''(t)\sin^2 t\big]\,\frac{1}{r(t)}. \end{aligned} \tag{15}$$

Also, from (15) we have

$$\frac{dr}{dt}(t) = \frac{2p(t)p'(t)}{r(t)}\cos^2 t + O(t^{-\alpha-2}) \text{ as } t \to \infty. \tag{16}$$

Since $\frac{dr}{dt}(t) = f'(\varphi) \cdot \frac{d\varphi}{dt}(t)$ and since by (12) we have $\frac{d\varphi}{dt}(t) \simeq 1$ as $t \to \infty$, there exists $C_0 > 0$ and $C_1 > C_0$ such that for all $\varphi \in [\varphi_0, \infty)$,

$$|f'(\varphi)| \leq C_0 t^{-\alpha-1} \leq C_1 \varphi^{-\alpha-1}. \tag{17}$$

Step 4. (Checking condition (3).) Using (10) and [6, Lemma 6], we obtain

$$\tan\varphi(t) = -(\cot t + O(t^{-1})) = -\cot(t + O(t^{-1})) = \tan(t + \frac{\pi}{2} + O(t^{-1}))$$

as $t \to \infty$. Since the function $\varphi(t)$ is continuous and $O(t^{-1}) < \pi$ for t large enough, then there exists $k \in \mathbb{Z}$ such that

$$\varphi(t) = (t + \frac{\pi}{2} + k\pi) + O(t^{-1}) \text{ as } t \to \infty.$$

From the definition of $\varphi(t)$ we conclude that we may take without loss of generality $k = 0$. Finally, we get

$$\varphi(t) = \left(t + \frac{\pi}{2}\right) + O(t^{-1}) \text{ as } t \to \infty. \tag{18}$$

Also, notice that from (10) it follows that the function φ is of class C^2.

Step 5. (Checking condition (6).) From (12) it follows that $\frac{d\varphi}{dt}(t) > 0$ for all t large enough, so then the function $\varphi(t)$ is increasing. As $\varphi(t)$ is continuous, we conclude that for all φ large enough there exists the inverse function $t = t(\varphi)$ of the function $\varphi = \varphi(t)$ and

$$t(\varphi) = \left(\varphi - \frac{\pi}{2}\right) + O(\varphi^{-1}) \text{ as } \varphi \to \infty.$$

Also, $\frac{d\varphi}{dt}(t) > 0$ for all t large enough means that there exists $t_1 \geq t_0$ such that $\frac{d\varphi}{dt}(t) > 0$ for all $t \geq t_1$. We define value $\varphi_1 = \varphi(t_1)$. Notice that we can take t_1 such that $\varphi_1 \geq \varphi_0$.

From (13) we obtain

$$r(t) = p(t)\sqrt{1 + \frac{p'^2(t)}{p^2(t)}\sin^2 t + \frac{p'(t)}{p(t)}\sin 2t}.$$

By Proposition 2 we conclude that for fixed $\triangle\varphi > 1$ we have

$$f(\varphi) - f(\varphi + \triangle\varphi) = r(t(\varphi)) - r(t(\varphi + \triangle\varphi)) \geq k_1 \varphi^{-\alpha-1}, \tag{19}$$

provided φ is large enough. Moreover, by careful examination of the proof of Proposition 2, we conclude that the statement (19) uniformly

holds for every $\triangle\varphi$ from a bounded interval whose lower bound is greater than 1, also provided φ is large enough.

Step 6. (Γ' is a spiral near the origin.) Now we can prove that Γ' is a spiral near the origin, that is, $f(\varphi)$ satisfies condition (1) near the origin. First, from (14) it follows that $f(\varphi) \to 0$ as $\varphi \to \infty$. Second, from (19) it follows that $f(\varphi)$ is radially decreasing for all φ large enough, that is, there exists $\overline{\varphi}_1 \geq \varphi_1$ such that $f|_{[\overline{\varphi}_1,\infty)}$ is radially decreasing.

Step 7. (Box dimension is invariant with respect to taking t_0 and φ_0 sufficiently large.) First, we define $\overline{t}_1$ to be such that $\varphi(\overline{t}_1) = \overline{\varphi}_1$. Notice that $\overline{t}_1$ is well-defined and $\overline{t}_1 \geq t_1$. As $p(t) > 0$, from (13) and the definition of $x(t)$ and $\dot{x}(t)$ it follows that $r(t) > 0$, that is, $r(t)$ is strictly positive function. That means there exists constant $m_1 > 0$ such that for all $t \in [t_0, \overline{t}_1]$ it holds

$$r(t) > m_1. \tag{20}$$

Notice that $\overline{\varphi}_1 \geq \varphi_1 \geq \varphi_0$. From (14) it follows that $r(t) \to 0$ as $t \to \infty$ so there exists $\overline{t}_2 \geq \overline{t}_1$ such that for all $t \in [\overline{t}_2, \infty)$ it holds

$$r(t) < m_1. \tag{21}$$

We define value $\overline{\varphi}_2 = \varphi(\overline{t}_2)$. From (20) and (21) we conclude that

$$\Gamma'|_{[t_0,\overline{t}_1]} \bigcap \Gamma'|_{(\overline{t}_2,\infty)} = \emptyset. \tag{22}$$

As $f|_{[}\overline{\varphi}_1, \infty)$ is radially decreasing and $\varphi'(t) > 0$ for all $t \in [\overline{t}_1, \infty)$ it follows that $\Gamma'|_{(\overline{t}_1,\infty)}$ does not have self intersections, so

$$\Gamma'|_{[\overline{t}_1,\overline{t}_2]} \bigcap \Gamma'|_{(\overline{t}_2,\infty)} = \emptyset. \tag{23}$$

Finally, from (22) and (23) we have $\Gamma'|_{[t_0,\overline{t}_2]} \bigcap \Gamma'|_{(\overline{t}_2,\infty)} = \emptyset$. Now, we can apply Lemma 1, so we can always take t and φ large enough without changing the box dimension of Γ'. Informally speaking, we can always remove any rectifiable part from the beginning of Γ'.

Step 8. (Checking wavyness condition (2).) By factoring (15), we get

$$\frac{dr}{dt}(t) = \left(1 + \frac{p'(t)}{p(t)}\tan t\right)\left(1 + \frac{p''(t)}{2p'(t)}\tan t\right)\frac{2p(t)p'(t)}{r(t)}\cos^2 t. \tag{24}$$

By Lemma 4 and Remark 7, see below, and using (8) and (9), there exists $k_0 \in \mathbb{N}_0$ such that the equations

$$\tan t = -\frac{p(t)}{p'(t)}, \qquad \tan t = -\frac{2p'(t)}{p''(t)},$$

have unique solutions $\hat{t}_{2k}$ and t_{2k-1}, respectively, in intervals $((k-1+k_0)\pi, (k+k_0)\pi - \frac{\pi}{2})$, for each $k \in \mathbb{N}$, since

$$-\frac{p(t)}{p'(t)} \sim \frac{t}{\alpha}, \qquad -\frac{2p'(t)}{p''(t)} \sim \frac{2t}{\alpha+1} \qquad \text{as } t \to \infty.$$

Without loss of generality we can take $t_{2k-1} < \hat{t}_{2k}$. It is easy to see from (24) that $\frac{dr}{dt}(t)$ is positive between these solutions. Notice that due to Lemma 1 without loss of generality we can take $t_0 \in (k_0\pi, t_1)$ for which $r'(t_0) \leq 0$. As $\frac{d\varphi}{dt}(t) > 0$ for all t large enough, from $\frac{dr}{dt}(t) = f'(\varphi) \cdot \frac{d\varphi}{dt}(t)$ it follows that $f'(\varphi) > 0$ on the set $\bigcup_{k=1}^{\infty}(\varphi_{2k-1}, \hat{\varphi}_{2k})$ where $\varphi_{2k-1} = \varphi(t_{2k-1})$ and $\hat{\varphi}_{2k} = \varphi(\hat{t}_{2k})$. This implies that the function $r = f(\varphi)$ is increasing for some φ, so we can not apply Theorem 2 directly. From (8) and using inequalities $1 < 2/(\alpha+1) < 1/\alpha$, we see that the subsequence $(t_{2k-1})_{k\geq 1}$ satisfies $\hat{t}_{2k} - t_{2k-1} < \pi/12$ and $t_{2k+1} - \hat{t}_{2k} < 3\pi/2$ for every $k \in \mathbb{N}$.
It is easy to see that the subsequence $(t_{2k+1})_{k\geq 0}$ has exactly the same values as the subsequence $(t_{2k+1})_{k\geq 0}$ in Definition 9, defined for the function $r(t)$. From before it follows that $t_{2k+1} \to \infty$ as $k \to \infty$, the subsequence $(t_{2k+1})_{k\geq 0}$ belongs to the sequence (t_n) from Definition 9 and the sequence (t_n) is increasing by its inductive definition, we see that the sequence (t_n) satisfies condition (2)(i).
On the other hand for t_{2k+2}, $k \in \mathbb{N}_0$, from Definition 9 it follows that $t_{2k+1} < \hat{t}_{2k+2} < t_{2k+2}$. From the definition of sequence (t_n) in Definition 9 it follows that $r(t_{2k+2}) = r(t_{2k+1})$, so

$$r(\hat{t}_{2k+2}) - r(t_{2k+2}) = r(\hat{t}_{2k+2}) - r(t_{2k+1}), \quad k \in \mathbb{N}_0. \tag{25}$$

Using (16) we conclude that there exist constants $C_3 > 0$ and $C_4 > C_3$ such that

$$\begin{aligned} r(\hat{t}_{2k+2}) - r(t_{2k+1}) &= \int_{t_{2k+1}}^{\hat{t}_{2k+2}} r'(t)\,dt \\ &\leq \frac{\pi}{12} \sup_{t\in[t_{2k+1},\hat{t}_{2k+2}]} r'(t) \leq C_3 t_{2k+1}^{-\alpha-2} \leq C_4 \hat{t}_{2k+2}^{-\alpha-2}, \end{aligned} \tag{26}$$

for every $k \in \mathbb{N}_0$.
Now, we first define $\varphi_{2k+2} = \varphi(t_{2k+2})$, for every $k \in \mathbb{N}_0$. Assume that $\varphi_{2k+2} - \hat{\varphi}_{2k+2} > 1$. Now from Proposition 2 (notice that $\varphi_{2k+2} - \hat{\varphi}_{2k+2} > 1$ and has the uniform upper bound with respect to k), similary as in

(19), there exist constants $K_1 > 0$ and $0 < K_2 < K_1$ such that

$$\begin{aligned} r(\hat{t}_{2k+2}) - r(t_{2k+2}) &= f(\hat{\varphi}_{2k+2}) - f(\varphi_{2k+2}) \\ &= f(\hat{\varphi}_{2k+2}) - f(\hat{\varphi}_{2k+2} + (\varphi_{2k+2} - \hat{\varphi}_{2k+2})) \\ &\geq K_1 \hat{\varphi}_{2k+2}^{-\alpha-1} \geq K_2 \hat{t}_{2k+2}^{-\alpha-1}, \end{aligned} \tag{27}$$

for k sufficiently large.
Substituting (27) and (26) in (25) we finally get

$$K_2 \hat{t}_{2k+2}^{-\alpha-1} \leq C_4 \hat{t}_{2k+2}^{-\alpha-2},$$

for k sufficiently large, which gives the contradiction. It follows that $\varphi_{2k+2} - \hat{\varphi}_{2k+2} \leq 1$, which gives $t_{2k+2} - \hat{t}_{2k+2} \leq \frac{\pi}{3}$, for k sufficiently large, because $\frac{\pi}{3} > 1$. Now $t_{2k+2} - t_{2k+1} = (t_{2k+2} - \hat{t}_{2k+2}) + (\hat{t}_{2k+2} - t_{2k+1}) < \frac{\pi}{3} + \frac{\pi}{12} = \frac{5\pi}{12}$, and this means that the sequence (t_n) satisfies condition (2)(ii).
Notice that analogously as in (26), using (16), as $\operatorname*{osc}_{t \in [t_{2k+1}, t_{2k+2}]} r(t) = r(\hat{t}_{2k+2}) - r(t_{2k+1})$, we can show that (t_n) satisfies condition (2)(iii).
Finally we conclude that the sequence (t_n) satisfies wavyness condition (2), so $r(t)$ is a wavy function and Γ' is a wavy spiral near the origin.

Step 9. (Final conclusion.) From previous steps we see directly that all of the assumptions of Theorem 3 are fulfilled. Using Theorem 3 we prove the first claim of our theorem, that $\dim_B \Gamma' = \frac{2}{1+\alpha}$.

(ii) To prove that Γ is a wavy spiral near the origin notice that steps 1–8 also hold for $\alpha > 1$.
To prove the rectifiability for $\alpha > 1$, from (14), (12) and (16) we have that there exist positive constants C_5, M_1 and C_6 such that for every $t \in [t_0, \infty)$ it holds

$$r(t) \leq C_5 t^{-\alpha}, \quad \varphi'(t) \leq M_1, \quad r'(t) \leq C_6 t^{-\alpha-1}.$$

Therefore

$$\begin{aligned} l(\Gamma) &= l(\Gamma') = \int_{t_0}^{\infty} \sqrt{(r(t)\varphi'(t))^2 + (r'(t))^2}\, dt \\ &\leq \int_{t_0}^{\infty} \sqrt{M_1^2 C_5^2 t^{-2\alpha} + C_6^2 t^{-2\alpha-2}}\, dt \leq M_2(t_0) \int_{t_0}^{\infty} |t|^{-\alpha}\, dt < \infty. \end{aligned}$$

$\square$

4. Chirps generated by spirals

Now we study some a converse of Theorem 4, where we obtain the box dimension of a chirp from the corresponding spiral. We begin with

a thorem concerning the box dimension of the graph of a generalized (α, β)-chirp.

Theorem 5. (Box dimension and Minkowski content of the graph of a generalized (α, β)-*chirp*) *Let* $y(x) = p(x)S(q(x))$, $x \in I = (0, c]$, $c > 0$. *Let the functions* $p(x)$, $q(x)$ *and* $S(t)$ *satisfy the following assumptions:*

$$p \in C(\bar{I}) \cap C^1(I),\ q \in C^1(I),\ S \in C^1(\mathbb{R}), \tag{28}$$

The function $S(t)$ *is assumed to be a* $2T$*-periodic real function defined on* $\mathbb{R}$ *such that*

$$\begin{cases} S(a) = S(a+T) = 0 \text{ for some } a \in \mathbb{R}, \\ S(t) \neq 0 \text{ for all } t \in (a, a+T) \cup (a+T, a+2T), \end{cases} \tag{29}$$

where T *is a positive real number and* $S(t)$ *alternately changes sign on intervals* $(a + (k-1)T, a + kT)$, $k \in \mathbb{N}$. *Without loss of generality, we take* $a = 0$. *Let us suppose that* $0 < \alpha \le \beta$ *and:*

$$p(x) \simeq_1 x^{\alpha} \quad as \quad x \to 0, \tag{30}$$

$$q(x) \simeq_1 x^{-\beta} \quad as \quad x \to 0. \tag{31}$$

Then $y(x)$ *is* d*-dimensional fractal oscillatory near the origin, where* $d = 2 - \frac{\alpha+1}{\beta+1}$. *Moreover,* $\dim_B(G(y)) = d$ *and* $G(y)$ *is Minkowski nondegenerate.*

Theorem 5 is a new version of [4, Theorems 5 and 6]. In Theorem 5 we do not need any assumptions on the curvature function of $y(x) = p(x)S(q(x))$, as it was needed in [4]. Before proving Theorem 5, we shall cite a new criterion for fractal oscillations of a bounded continuous function and after that we continue with two propositions concerning the properties of functions p, q and S.

Theorem 6. ([14, Theorem 2.1.]) *Let* $y \in C^1((0, T])$ *be a bounded function on* $(0, T]$. *Let* $s \in [1, 2)$ *be a real number and let* (a_n) *be a decreasing sequence of consecutive zeros of* $y(x)$ *in* $(0, T]$ *such that* $a_n \to 0$ *when* $n \to \infty$ *and let there exist constants* c_1, c_2, ε_0 *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *we have:*

$$c_1 \varepsilon^{2-s} \le \sum_{n \ge k(\varepsilon)} \max_{x \in [a_{n+1}, a_n]} |y(x)|(a_n - a_{n+1}), \tag{32}$$

$$a_{k(\varepsilon)} \sup_{x \in (0, a_{k(\varepsilon)}]} |y(x)| + \varepsilon \int_{a_{k(\varepsilon)}}^{a_1} |y'(x)| dx \le c_2 \varepsilon^{2-s}, \tag{33}$$

where $k(\varepsilon)$ *is an index function on* $(0, \varepsilon_0]$ *such that*

$$|a_n - a_{n+1}| \le \varepsilon \quad \textit{for all} \quad n \ge k(\varepsilon) \quad \textit{and} \quad \varepsilon \in (0, \varepsilon_0).$$

Then $y(x)$ is fractal oscillatory near $x = 0$ with $\dim_B G(y) = s$.

We remark that the claim of Theorem 6 is true if we substitute a_1, appearing in (33) by a_{k_0}, where k_0 is a fixed natural number.

Proposition 3. *Assume that the functions $p(x)$ and $q(x)$ satisfy conditions (28), (30) and (31). Then there exist $\delta_0 > 0$ and positive constants C_1 and C_2 such that:*

$$C_1 x^\alpha \le p(x) \le C_2 x^\alpha, \; C_1 x^{\alpha-1} \le p'(x) \le C_2 x^{\alpha-1}, \tag{34}$$

$$C_1 x^{-\beta} \le q(x) \le C_2 x^{-\beta}, \; C_1 x^{-\beta-1} \le -q'(x) \le C_2 x^{-\beta-1}, \tag{35}$$

for all $x \in (0, \delta_0]$. Furthermore, there exists the inverse function q^{-1} of the function q defined on $[m_0, \infty)$, where $m_0 = q(\delta_0)$, and it holds:

$$q^{-1}(t) \simeq_1 t^{-1/\beta} \quad \textit{as} \quad t \to \infty, \tag{36}$$

$$C_1 t^{-\frac{1}{\beta}-1}(t-s) \le q^{-1}(s) - q^{-1}(t) \le C_2 s^{-\frac{1}{\beta}-1}(t-s), \quad m_0 \le s < t. \tag{37}$$

Proposition 4. *For any function $S(t)$ satisfying (29), and for any function $q(x)$ with properties (28) and (31), we have:*

(i) *$S(kT) = 0, \; k \in \mathbb{N}$.*

(ii) *Let $a_k = q^{-1}(kT)$ and $s_k = q^{-1}(t_0 + kT)$, $k \in \mathbb{N}$, where $t_0 \in (0, T)$ is arbitrary. Then there exist $k_0 \in \mathbb{N}$ and $c_0 > 0$ such that $a_k \in (0, \delta_0]$ $y(a_k) = 0$, $s_k \in (a_{k+1}, a_k)$ for all $k \ge k_0$, $a_k \searrow 0$ as $k \to \infty$, $a_k \simeq k^{-1/\beta}$ as $k \to \infty$, and*

$$\max_{x \in [a_{k+1}, a_k]} |y(x)| \ge c_0 (k+1)^{-\alpha/\beta} \quad \textit{for all } k \ge k_0, \; c_0 > 0. \tag{38}$$

(iii) *There exists $\varepsilon_0 > 0$ and a function $k : (0, \varepsilon_0) \to \mathbb{N}$ such that*

$$\frac{1}{T}\left(\frac{\varepsilon}{TC_2}\right)^{-\frac{\beta}{\beta+1}} \le k(\varepsilon) \le \frac{2}{T}\left(\frac{\varepsilon}{TC_2}\right)^{-\frac{\beta}{\beta+1}}. \tag{39}$$

In particular,

$$C_1 T((k+1)T)^{-\frac{1}{\beta}-1} \le a_k - a_{k+1} \le \varepsilon,$$

for all $k \ge k(\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$.

Proofs of Propositions 3 and 4 are provided in the Appendix.

Proof of Theorem 5. First we check inequality (32). By Proposition 4 we have:

$$\begin{aligned}\sum_{k\geq k(\varepsilon)} \max_{x\in[a_{k+1},a_k]} |y(x)|(a_k - a_{k+1}) \geq & c\sum_{k=k(\varepsilon)+1}^{\infty} (k+1)^{-\frac{\alpha+\beta+1}{\beta}} \\ =& c\sum_{k=k(\varepsilon)}^{\infty} (k)^{-\frac{\alpha+\beta+1}{\beta}} = ca,\end{aligned}$$

where the series $a = \sum_{k=k(\varepsilon)}^{\infty} (k)^{-\frac{\alpha+\beta+1}{\beta}}$ is convergent, because of $\frac{\alpha+\beta+1}{\beta} > 1$. Then using the inequality $(\frac{1}{k(\varepsilon)})^{\frac{\alpha+\beta+1}{\beta}-1} < 1$ and (39) we obtain

$$\begin{aligned}\sum_{k\geq k(\varepsilon)} \max_{x\in[a_{k+1},a_k]} |y(x)|(a_k - a_{k+1}) &\geq ca \geq ca(\frac{1}{k(\varepsilon)})^{\frac{\alpha+\beta+1}{\beta}-1} \\ &\geq c_1\varepsilon^{\frac{\alpha+1}{\beta+1}} = c_1\varepsilon^{2-(2-\frac{\alpha+1}{\beta+1})},\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$. By [14, Lemma 2.1.] this implies

$$0 < \mathcal{M}_*^d(G(y)) \quad \text{and} \quad \underline{\dim}_B G(y) \geq d,$$

where $G(y)$ is the graph of y and $d = 2 - \frac{\alpha+1}{\beta+1}$. Now we check inequality (33). From (30) and (31) it follows that

$$|y'(x)| = |p'(x)S(q(x)) + p(x)q'(x)S'(q(x))| \leq cx^{\alpha-\beta-1},$$

which holds near $x = 0$, where

$$c = \max\{\max_{x\in[0,2T]} |S(t)|, \max_{x\in[0,2T]} |S'(t)|\}.$$

By Proposition 4 we have:

$$a_{k(\varepsilon)} \sup_{x\in(0,a_{k(\varepsilon)}]} |y(x)| + \varepsilon\int_{a_{k(\varepsilon)}}^{a_{k_0}} |y'(x)|dx \leq c\varepsilon^{\frac{\alpha+1}{\beta+1}} + \varepsilon[a_{k_0}^{\alpha-\beta} + a_{k(\varepsilon)}^{\alpha-\beta}] \leq c_2\varepsilon^{\frac{\alpha+1}{\beta+1}}$$

for all $\varepsilon \in (0, \varepsilon_0)$. By [14, Lemma 2.2.] it follows that

$$\mathcal{M}^{*d}(G(y)) < \infty \quad \text{and} \quad \overline{\dim}_B G(y) \leq d = 2 - \frac{\alpha+1}{\beta+1},$$

Finally, combining the obtained results, we have that the graph $G(y)$ is Minkowski nondegenerate, and

$$\dim_B G(y) = 2 - \frac{\alpha+1}{\beta+1} = d.$$

□

Now we can state a spiral-chirp comparison.

Theorem 7. (Spiral-chirp comparison) *Let $\alpha \in (0,1)$, and assume that $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, is a function of class C^2, such that the planar curve $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ is a spiral $r = f(\varphi)$, $\varphi \in (\varphi_0, \infty)$, $\varphi_0 > 0$, in polar coordinates, near the origin, such that $f(\varphi) \simeq_1 \varphi^{-\alpha}$, as $\varphi \to \infty$, and $\dot{\varphi}(t) \simeq 1$, as $t \to \infty$, where $\varphi(t)$ is a function of class C^1 defined by $\tan \varphi(t) = \frac{\dot{x}(t)}{x(t)}$. Define $X(\tau) = x(1/\tau)$. Then $X = X(\tau)$ is $(\alpha, 1)$-chirp-like function, and*

$$\dim_{osc}(x) := \dim_B G(X) = \frac{3-\alpha}{2},$$

where $G(X)$ is graph of the function X. Furthermore, $G(X)$ is Minkowski nondegenerate.

Proof. Let us write the function $X(\tau)$ in the form

$$X(\tau) = p(\tau) \cos q(\tau), \quad \tau \in (0, \frac{1}{t_0}],$$

where

$$p(\tau) = f(\varphi(\frac{1}{\tau})), \quad q(\tau) = \varphi(\frac{1}{\tau}).$$

The function $p(\tau)$ is increasing near $\tau = 0$ since $\frac{1}{\tau}$ is decreasing, $\varphi(t)$ is increasing and $f(\varphi)$ is decreasing near $\varphi = \infty$. Furthermore, $p \in C([0, 1/t_0])$ since $p(0) = \lim_{\tau \to 0} f(\varphi(1/\tau)) = 0$, by noting that $\dot{\varphi} \simeq 1$ implies $\varphi(t) \to \infty$ as $t \to \infty$. Now, we shall exploit Theorem 5, by checking that its assumptions are satisfied with $S(q) = \cos q$ and $\beta = 1$. The functions φ, p and q have the following properties:

$$\varphi(t) \simeq t \quad \text{as} \quad t \to \infty \quad \text{or} \quad \varphi(\frac{1}{\tau}) \simeq \frac{1}{\tau} \quad \text{as} \quad \tau \to 0,$$

$$\begin{aligned} p(\tau) &\simeq_1 \tau^{\alpha} \quad \text{as} \quad \tau \to 0, \\ q(\tau) &\simeq_1 \frac{1}{\tau} \quad \text{as} \quad \tau \to 0, \\ q^{-1}(t) &\simeq \frac{1}{t} \quad \text{as} \quad t \to \infty. \end{aligned}$$

The function q is decreasing near the origin, thus q^{-1} exists for t large enough. We see that all conditions of Theorem 5 are fulfilled. □

Remark 3. Theorem 7 is a new version of [15, Theorem 4]. If we compare Theorems 4 and 7 in terms of conditions, then we see that Theorem 7 requires derivatives of lower order than Theorem 4. Phase plane gives us already the information about the first derivative.

The following result shows that rectifiable spirals generate rectifiable chirp-like functions.

Theorem 8. (Rectifiability of a chirp generated by a rectifiable spiral) *Let $\alpha > 1$, and assume that $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, is a function of class C^2 such that the planar curve $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ is a rectifiable spiral $r = f(\varphi)$, $\varphi \in (\varphi_0, \infty)$, $\varphi_0 > 0$ in polar coordinates, near the origin, such that $f(\varphi) \simeq_1 \varphi^{-\alpha}$, as $\varphi \to \infty$, $|f''(\varphi)| \le C\varphi^{-\alpha-2}$ and $\dot{\varphi}(t) \simeq 1$ as $t \to \infty$, where $\varphi(t)$ is a function of class C^1 defined by $\tan \varphi(t) = \frac{\dot{x}(t)}{x(t)}$. Define $X(\tau) = x(1/\tau)$. Then $X = X(\tau)$ is $(\alpha, 1)$-chirp-like rectifiable function near the origin.*

To prove the theorem we shall use the following two lemmas.

Lemma 2. *Let $F, G \in C^1(I)$, where I is an open interval in $\mathbb{R}$, and assume that $\inf F' > \sup G'$. Then the equation $F(z) = G(z)$ has at most one solution.*

Proof. Suppose that there are two different solutions z_1 and z_2. Then applying the mean-value theorem to $F(z_1) - F(z_2) = G(z_1) - G(z_2)$, we obtain that there exist $\tilde{z}_1$ and $\tilde{z}_2$ such that $F'(\tilde{z}_1) = G'(\tilde{z}_2)$. Therefore, $\inf F' \le \sup G'$. This contradicts the condition $\inf F' > \sup G'$. $\square$

Lemma 3. *Let $F \in C^1(0, \infty)$ be such that $F(z) \sim az$ as $z \to \infty$ for some $a < 0$. Assume that $\inf F' > -\infty$. Then there exists a nonnegative integer k_0 such that for each $k \ge k_0$ the equation $\cot z = F(z)$ possesses the unique solution in $J_k = (k\pi, (k+1)\pi)$.*

Proof. Since $F(z)$ is continuous and $F(z) \sim az$ as $z \to \infty$, and $\cot z$ restricted to J_k is continuous function onto $\mathbb{R}$, it follows that the equation $\cot z = F(z)$ possesses at least one solution z_k on each interval J_k. We have to show that the solution is unique on each J_k for all k large enough.
Since $m = \inf F' > -\infty$, there exists $s_0 \in (\pi/2, \pi)$ sufficiently close to π such that $\cot'(s_0) = -(\sin s_0)^{-2} < m$. The condition $F(z) \sim az$ implies that, given any fixed $b \in (a, 0)$, there exists $M = M(b) > 0$ such that $F(z) < bz$ for all $z \ge M$. Let us fix any such b.
Let k_0 be a nonnegative integer such that $b(k_0\pi) < \cot s_0$. It suffices to take $k_0 > (b\pi)^{-1} \cot s_0$. Taking k_0 still larger, we can achieve that $k_0\pi \ge M$. Hence, for $z \ge k_0\pi$ we have $F(z) < bz$. In particular,

$$F(z) < bz \le b(k_0\pi) < \cot s_0.$$

Since for $z \ge k_0\pi$ we have $F(z) < \cot s_0$, while $\cot z \ge \cot s_0$ for each $z \in J_k \setminus I_k$, where $I_k = (k\pi + s_0, (k+1)\pi)$, then all solutions of equation $F(z) = \cot z$ for $z \ge k_0\pi$ are contained in $\cup_{k \ge k_0} I_k$.

Let us define $G(z) = \cot z$, and consider the equation $F(z) = G(z)$ on I_k for any $k \geq k_0$. We have

$$\sup_{I_k} G' = \cot'(k_0\pi + s_0) = -(\sin s_0)^{-2} < \inf_{(0,\infty)} F' \leq \inf_{I_k} F'.$$

The unique solvability of $F(z) = G(z)$ on I_k follows from Lemma 2. The equation is uniquely solvable on J_k as well, since there are no solutions in $J_k \setminus I_k$. $\square$

Remark 4. The condition $F(z) \sim az$ as $z \to \infty$ in Lemma 3 can be weakened. It suffices to assume that $F(z) < bz$ for some $b < 0$ and for all z large enough.

Remark 5. The condition $\inf F' > -\infty$ in Lemma 3 cannot be dropped. To see this, we construct a function $y = F(z)$ by means of a sequence of lines $y = b_n z$, where $a < b_n < 0$ and $b_n \to a$ as $n \to \infty$. We first construct a continuous function F_0 such that on $J_k' = (k\pi, (k+1)\pi]$,

$$F_0(z) = \begin{cases} b_k z, & \text{for } z \in (k\pi, z_k], \\ \cot z, & \text{for } z \in (z_k, v_k], \\ b_{k+1} z, & \text{for } z \in (v_k, (k+1)\pi], \end{cases}$$

where z_k and v_k are respective solutions of equations $\cot z = b_k z$ and $\cot b_{k+1} v = b_{k+1} v$ in J_k. The function F_0 is of class C^1 everywhere in $(0,\infty)$ except at the points z_k and v_k. We can perform its smoothing in sufficiently small neighborhoods of these points, in order to get a function $F \in C^1(0,\infty)$. It is clear that $F(z) \sim az$ as $z \to \infty$ and $\inf F' = -\infty$. But $F(z) = \cot z$ possesses infinitely many solutions on each interval I_k.

Remark 6. Assume that

$$F(z) = \frac{f(z)}{f'(z)},$$

where $f \in C^2(0,\infty)$. (a) The condition $\inf F' > -\infty$ is equivalent to $f(z)f''(z) \leq Cf'^2(z)$, where C is a positive constant. (b) The condition $F(z) < bz$ for z sufficiently large, where b is a negative constant (see Remark 4), is satisfied if for all z large enough we have $f(z) \geq az^{-\alpha}$ and $f'(z) \geq a_1 z^{-\alpha-1}$, where $a > 0$ and $a_1 < 0$ are constans. It suffices to take $b \in (a/a_1, 0)$.

A variation of Lemma 3 is the following lemma.

Lemma 4. *Let $F \in C^1(0,\infty)$ be such that $F(z) \sim az$ as $z \to \infty$ for some $a > 0$. Assume that $\sup F' < \infty$. Then there exists a nonnegative integer k_0 such that for each $k \geq k_0$ the equation $\tan z = F(z)$ possesses the unique solution in $J_k = ((k-1/2)\pi, (k+1/2)\pi)$.*

Remark 7. The condition $F(z) \sim az$ as $z \to \infty$ for $a > 0$ in Lemma 4 can be weakened by assuming that $F(z) > az$ for some $a > 0$ and for all z large enough. If $F(z)$ has the form $F(z) = \frac{f(z)}{f'(z)}$, where $f \in C^2(0,\infty)$, the condition $\sup F' < \infty$ is equivalent to $f(z)f''(z) \geq Cf'^2(z)$, where C is positive constant. Also, in that case, the condition $F(z) > az$ for z large enough is satisfied if for all z large enough we have $f(z) \geq a_1 z^{-\alpha}$ and $f'(z) \leq a_2 z^{-\alpha-1}$, where a_1 and a_2 are positive constants. It suffices to take $a \in (0, \frac{a_1}{a_2})$.

Proof of Theorem 8. We can write the function $X(\tau)$ in the form $X(\tau) = p(\tau)\cos q(\tau)$, where $p(\tau) = f(\varphi(1/\tau)) \simeq \tau^\alpha$, $p'(\tau) \simeq \tau^{\alpha-1}$, $q(\tau) = \varphi(1/\tau) \simeq \tau^{-1}$, $q'(\tau) \simeq -\tau^{-2}$ as $\tau \to 0$. It follows that X is an $(\alpha,1)$-chirp-like function. Using the assumptions of the theorem, for the function

$$F(t) := \frac{pq'}{p'}(q^{-1}(t)) = \frac{f(t)}{f'(t)}$$

we have $F(t) \simeq -t$ as $t \to \infty$, and $\frac{f(t)f''(t)}{f'^2(t)} < C$, for t large enough, $C > 0$. Then there exists $k_0 \in \mathbb{N}$ such that the equation $\cot q(t) = F(q(t)) = \frac{p(\tau)q'(\tau)}{p'(\tau)}$ has the unique solution $s_k \in (a_{k+1}, a_k)$ where $a_{k+1} = q^{-1}((2k+1)\frac{\pi}{2}$ and $a_k = q^{-1}((2k-1)\frac{\pi}{2})$ for all $k \geq k_0$, see Lemma 3 and Remark 4. These solutions are just points of local extrema of $X(\tau)$ on (a_{k+1}, a_k), $k \geq k_0$. The sequence $(a_k)_{k\geq 1}$ of zero points of X on $(0, 1/t_0]$ is decreasing. Hence the sequence (s_k) of consecutive points of local extrema of X is also decreasing. We have that $a_k = q^{-1}((2k-1)\frac{\pi}{2}) \simeq k^{-1}$ as $k \to \infty$. So the same is true also for s_k, i.e. $s_k \simeq k^{-1}$ as $k \to \infty$, and we also have $|X(s_k)| \leq p(s_k) \leq Cs_k^\alpha \leq C_1 k^{-\alpha}$. This implies that

$$\sum_{k=k_0}^{\infty} |X(s_k)| \leq C_1 \sum_{k=k_0}^{\infty} k^{-\alpha} < \infty \tag{40}$$

for $\alpha > 1$. The length of the graph $G(X)$ is defined by

$$\text{length}(G(X)) := \sup \sum_{i=1}^{m} \|(t_i, X(t_i)) - (t_{i-1}, X(t_{i-1}))\|_2,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_m = 1/t_0$ of the interval $[0, 1/t_0]$ and where $\|.\|_2$ denotes the Euclidean

norm in $\mathbb{R}^2$. Using [12, Lemma 3.1.], it follows that length$(G(X)) \leq 2\sum_k |X(s_k)| + 1/t_0$. Then X is rectifiable due to (40). $\square$

Remark 8. Theorem 7 can be applied to planar systems with pure imaginary pair of eigenvalues, because the normal forms of such systems satisfy the conditions of Theorem 7, see [19, Theorem 9]. Also, Theorem 7 can be applied to related second order differential equations.

5. Appendix

Proof of Proposition 2. From the assumption that $p(t) \sim_1 t^{-\alpha}$ as $t \to \infty$ we have that for each $\varepsilon > 0$ there exist $\overline{t}_0 \geq t_0$ such that for all $t \geq \overline{t}_0$,

$$(1-\varepsilon)t^{-\alpha} < p(t) < (1+\varepsilon)t^{-\alpha},$$
$$(1-\varepsilon)\alpha t^{-\alpha-1} < -p'(t) < (1+\varepsilon)\alpha t^{-\alpha-1},$$
$$\frac{p'(t)}{p(t)} > -\frac{K_1\alpha}{t}, \quad \text{where } K_1 := \frac{1+\varepsilon}{1-\varepsilon}.$$

Then we have

$$r_{\min}(t) \leq r(t) \leq r_{\max}(t), \quad \text{for all} \quad t \in [\overline{t}_0, \infty),$$

where

$$\begin{aligned} r_{\min}(t) &:= p(t)\sqrt{1 - K_1\alpha t^{-1}}, \\ r_{\max}(t) &:= p(t)\sqrt{1 + K_1\alpha t^{-1} + K_1^2\alpha^2 t^{-2}} \end{aligned}$$

and without loss of generality we assume that $\overline{t}_0 > K_1\alpha$. Therefore

$$r(t(\varphi)) - r(t(\varphi + \triangle\varphi)) \geq r_{\min}(t_1) - r_{\max}(t_2) =$$
$$\frac{p^2(t_1)(1 - K_1\alpha t_1^{-1}) - p^2(t_2)(1 + K_1\alpha t_2^{-1} + K_1^2\alpha^2 t_2^{-2})}{p(t_1)\sqrt{1 - K_1\alpha t_1^{-1}} + p(t_2)\sqrt{1 + K_1\alpha t_2^{-1} + K_1^2\alpha^2 t_2^{-2}}},$$

where

$$t_1 = t(\varphi) = \varphi(1 + O(\varphi^{-1})), \; t_2 = t(\varphi + \triangle\varphi)) = \varphi(1 + O(\varphi^{-1})),$$

as $t \to \infty$. Now, using relations

$$p(t_1)\sqrt{1 - K_1\alpha t_1^{-1}} \simeq \varphi^{-\alpha}, \; p(t_2)\sqrt{1 + K_1\alpha t_2^{-1} + K_1^2\alpha^2 t_2^{-2}} \simeq \varphi^{-\alpha},$$

$$t_1^{-\alpha} \simeq \varphi^{-\alpha}, \; t_2^{-\alpha} \simeq \varphi^{-\alpha} \quad \text{as } \varphi \to \infty,$$

$$p(t_1)\sqrt{1 - K_1\alpha t_1^{-1}} + p(t_2)\sqrt{1 + K_1\alpha t_2^{-1} + K_1^2\alpha^2 t_2^{-2}} \leq 2C_1\varphi^{-\alpha},$$

$$p^2(t_1) - p^2(t_2) \geq 2(1-\varepsilon)^2\triangle\varphi\alpha\varphi^{-2\alpha-1}(1 + O(\varphi^{-1})),$$

$$-K_1\alpha(p^2(t_1)t_1^{-1}+p^2(t_2)t_2^{-1})\geq -2K_1\alpha(1+\varepsilon)^2\varphi^{-2\alpha-1}(1+O(\varphi^{-1})),$$

$$-K_1^2\alpha^2p^2(t_2)t_2^{-2}=O(\varphi^{-2\alpha-2}),$$

where $C_1>0$ is independent of φ, we have

$$r(t(\varphi))-r(t(\varphi+\triangle\varphi))\geq$$

$$\frac{\alpha(1-\varepsilon)^2}{C_1}\left[\triangle\varphi-\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^3\right]\varphi^{-\alpha-1}(1+O(\varphi^{-1})).$$

If $\triangle\varphi>1$, then choosing

$$\varepsilon<\frac{(\triangle\varphi)^{1/3}-1}{(\triangle\varphi)^{1/3}+1},$$

we obtain the claim. □

Proof of Proposition 3. From (30) and (31) inequalities (34) and (35) follow directly by definition. The function $q|_{(0,\delta_0]}$ is positive and decreasing, and its inverse function is defined on $[m_0,\infty)$. Relation (36) follows from (35) applying the well known formula for derivative of the inverse function. Then exploiting the mean value theorem and (36), we get (37). □

Proof of Proposition 4. The claim in (i) is evident. To prove (ii), it suffices to take $k_0\in\mathbb{N}$ such that $k_0T\geq m_0$. We shall prove inequality (38) only, because other facts are easy consequences of Proposition 3. From (30) we obtain that $p(x)$ is positive and increasing function near $x=0$, and we have

$$\begin{aligned}\max_{x\in[a_{k+1},a_k]}|y(x)|\geq & p(s_k)|S(q(s_k))|\\ \geq & cp(a_{k+1})\geq c_1(a_{k+1})^\alpha\geq c_0(k+1)^{-\frac{\alpha}{\beta}},\end{aligned}$$

for all $k\geq k_0$, where $c=\min\{|S(t_0)|,|S(t_0+T)|\}$, $c_1=cC_1$ and $c_0=cC_1^2$ are positive constants. Now we prove (iii). Let $\varepsilon>0$ and let $k(\varepsilon)\in\mathbb{N}$ be such that

$$k(\varepsilon)\geq\frac{1}{T}\left(\frac{\varepsilon}{TC_2}\right)^{-\frac{\beta}{\beta+1}}=c\varepsilon^{-\frac{\beta}{\beta+1}},\quad c=T^{-1}(TC_2)^{\frac{\beta}{\beta+1}}.$$

Let ε_0' be such that for all $0<\varepsilon\leq\varepsilon_0'$ it holds $k(\varepsilon)T\geq m_0=q(\delta_0)$. Further, for all $\varepsilon<c^{\frac{\beta+1}{\beta}}$ we have $2c\varepsilon^{-\frac{\beta}{\beta+1}}-c\varepsilon^{-\frac{\beta}{\beta+1}}>1$. So, there exists $k(\varepsilon)\in\mathbb{N}$ such that

$$1<c\varepsilon^{-\frac{\beta}{\beta+1}}\leq k(\varepsilon)\leq 2c\varepsilon^{-\frac{\beta}{\beta+1}},\quad\text{for all}\quad\varepsilon<c^{\frac{\beta+1}{\beta}}.$$

Let us take $\varepsilon_0 = \min\{\varepsilon_0', c^{\frac{\beta+1}{\beta}}\}$. Then we can find $k(\varepsilon) \in \mathbb{N}$ such that

$$c\varepsilon^{-\frac{\beta}{\beta+1}} \le k(\varepsilon) \le 2c\varepsilon^{-\frac{\beta}{\beta+1}}, \quad k(\varepsilon)T \ge m_0 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0).$$

Using (37), then for all $k \ge k(\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$ it holds

$$C_1 T((k+1)T)^{-\frac{1}{\beta}-1} \le a_k - a_{k+1} \le \varepsilon.$$

$\square$

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